

Gravitational Stability and Screening Effect from D Extra Timelike Dimensions

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abstract

We study $(3 + 1) + D$ dimensional spacetime, where D extra dimensions are timelike. Compactification of the D timelike dimensions leads to tachyonic Kaluza-Klein gravitons. We calculate the gravitational self-energies of massive spherical bodies due to the tachyonic exchange, discuss their stability, and find that the gravitational force is screened in a certain number of the extra dimensions. We also derive the exact relationship between the Newton constants in the full $4 + D$ dimensional spacetime with the D extra times and the ordinary Newton constant of our 4 dimensional world.

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1. Introduction

Extra dimensions have been studied in a variety of contexts. Especially in recent works [1] -[3] $4 + D$ dimensional theories, where the D extra dimensions are spacelike ones of the size $L_{(D)}$, are discussed in order to solve the hierarchy problem without using low-energy supersymmetry or technicolor. The Planck scale, $M_{Pl(4)}$, of the four dimensional theory is related with the one, $M_{Pl(4+D)}$, of the $4 + D$ dimensional theories as $M_{Pl(4)}^2 \sim L_{(D)}^D M_{Pl(4+D)}^{D+2}$. For $M_{Pl(4+D)} \sim 1\text{TeV}$ $L_{(D)}$ is of the order $10^{\frac{30}{D}-17}\text{cm}$, and then $L_{(2)}$ comes out to be a submillimeter for $D = 2$. When the number D of the extra dimensions increases, the corresponding scale, $L_{(D)}$, becomes smaller. So for $D \geq 2$ the gravitational force is allowed to feel these large extra dimensions without any conflict to experiments [1], but other standard model fields are not [3]. Consequently, our universe is a four dimensional wall in the $4 + D$ dimensional space, that is a 3-brane, and only gravitons can propagate in the bulk space, while other standard model fields live in the 3-brane and are not able to propagate in the extra dimensions [4] [5].

On the other hand there is a possibility of the existence of extra timelike dimensions, which is an interesting problem in its own right and has also been discussed in various contexts like string theories [6], brane theories [7], and so on [8]. The compactification of timelike curves gives rise to tachyons and causes the violation of causality and conserved probability, but, as discussed in Ref.[9], the existence of extra timelike dimensions may not be unphysical if the effects of the violation do not conflict with physical observations and experiments. Similarly Ref.[10] studied the Newtonian potential with the tachyon exchange in a simplified but unjustified treatment, and suggested that the real part of the gravitational self-energy of uniform massive body with a radius $R < 2\pi L$, where L is the size of extra timelike dimensions, is screened. As has been shown rigorously in our previous paper [11], for the gravitational self-energy of any spherical massive body the correct screening range is actually $R \leq \pi L$ contrary to the claim of Ref.[10].

In the following we shall concentrate on the timelike D extra dimensions. From the compactification of the D extra dimensions on the timelike circles of a radius L we obtain tachyonic Kaluza-Klein (KK) modes [9]. Let gravitons propagate in the extra dimensions, then up to spin factors their propagators are proportional to

$$-i \frac{1}{k_0^2 - \mathbf{k}^2 + \frac{n^2}{L^2} + i\epsilon}, \quad n^2 = \sum_{i=1}^D n_i^2 \quad (1.1)$$

where $n_i \in \mathbf{Z}$. Eq.(1.1) is the ordinary massless graviton propagator for $n = 0$, that is, for $(n_1, \dots, n_D) = (0, \dots, 0)$, while for $n \neq 0$ it gives the tachyonic KK graviton propagators. These tachyonic gravitons normally lead to the imaginary parts of the gravitational self-energies for massive bodies, thus causing their gravitational instability, and consequently, resulting in the violation of causality and conserved probability in related physical processes [9][10].

In this paper we let gravitons propagate in the D extra timelike dimensions which are compactified on the timelike circles of the radius L . Then we investigate the gravitational stability of massive bodies and the screening effect of the gravitational force due to the exchange of the KK mode tower of tachyonic gravitons. In Section 2 we calculate the gravitational self-energies of massive bodies with some typical mass densities which are spherically symmetric. And in Section 3 we discuss the gravitational stability conditions of the spherical massive bodies and report on the notable generic relations between the number D of the extra timelike dimensions and the gravitational stability. Section 4 is devoted to the screening effect of the gravitational force, whose realization is reported to again depend on the number D of the extra timelike dimensions. In Section 5 we discuss the complexity of the Newtonian potential in the D extra times, and derive the relation between the Newton constants of the full $4 + D$ dimensional spacetime and the Newton constant of the ordinary 4 dimensional world. In Section 6 we present comments and conclusions. We shall give some useful formulas in the appendix.

2. The self energy of spherical bodies

From Eq.(1.1) we obtain a gravitational potential between two unit mass point at distance d as

$$V(d) = -G_N \frac{1}{d} - \sum_{\substack{n=\sqrt{n_1^2+\dots+n_D^2} \neq 0 \\ -\infty < n_i < \infty}} G_N \frac{1}{d} \exp\left(i \frac{n}{L} d\right), \quad (2.1)$$

in the non-relativistic tree-level approximation, where G_N is the Newton constant. The first term is the contribution of ordinary massless gravitons and the second is the one of tachyonic KK gravitons, which leads to the imaginary parts of gravitational self-energies. Now, since $\lim_{n \rightarrow 0} G_N \frac{1}{d} \exp\left(i \frac{n}{L} d\right) = G_N \frac{1}{d}$, we can rewrite Eq.(2.1) as

$$V(d) = - \sum_{\substack{n=\sqrt{n_1^2+\dots+n_D^2} \\ -\infty < n_i < \infty}} G_N \frac{1}{d} \exp\left(i \frac{n}{L} d\right). \quad (2.2)$$

We remind the reader at this point that the complex potential used in Refs.[9][10] has a wrong sign in the phase factor, and that the correct sign of the phase together with the correct overall sign of the potential is crucially important in discussing the stability of matter from the point of view of its vanishing or explosion [9].

Now we consider spherically symmetric bodies of radii R with a mass density $\rho(r)$, where r is a radial coordinate, and calculate their gravitational self-energies. We integrate over the opening angle β between every two mass points of densities $\rho(r)$ and $\rho(l)$ which are apart from each other at the distance $d = \sqrt{r^2 + l^2 - 2rl \cos \beta}$, and then we find the self-energies of spherical bodies to become

$$E_D(R) = \sum_{\substack{n=\sqrt{n_1^2+\dots+n_D^2} \\ -\infty < n_i < \infty}} 8i\pi^2 G_N L \int_0^R dr \int_0^r dl \rho(r) \rho(l) \frac{rl}{n} f(n, r, l), \quad (2.3)$$

$$f(n, r, l) \equiv \exp\left(\frac{in}{L}(r+l)\right) - \exp\left(\frac{in}{L}(r-l)\right).$$

In order to calculate the summations we assume the discrete numbers $\{n_i\}$ as continuous parameters $\{x_i\}$ and replace the infinite summations in Eq.(2.3) with the integrations over those parameters. This replacement is a fairly good approximation and will become exact for $r, l \ll L$, that is, when the size of spherical bodies, R , is sufficiently smaller than the one of the extra dimensions, L . Then we obtain

$$\begin{aligned} E_D(R) &\approx 8i\pi^2 G_N L \int_0^R dr \int_0^r dl \rho(r) \rho(l) rl \int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_D \frac{f(\sqrt{x_1^2 + \cdots + x_D^2}, r, l)}{\sqrt{x_1^2 + \cdots + x_D^2}} \\ &= 8i\pi^2 G_N L \int_0^R dr \int_0^r dl \rho(r) \rho(l) rl \int d\Omega_{D-1} \int_0^{\infty} dt t^{D-2} f(t, r, l) \\ &= 8i\pi^2 G_N L S_D \int_0^R dr \int_0^r dl \rho(r) \rho(l) rl \int_0^{\infty} dt t^{D-2} f(t, r, l), \end{aligned} \quad (2.4)$$

where $\int d\Omega_{D-1}$ is the volume of $D-1$ dimensional unit sphere, which is $S_D = \frac{2\pi^{D/2}}{\Gamma(D/2)}$. At this point, by transforming back the integration over t in the last line of Eq.(2.4) to the infinite summation over discrete non-negative integers, the present approximate expression of the self-energies will become more accurate and come close to the starting formula, Eq.(2.3). We then obtain from Eq.(2.4)

$$\begin{aligned} E_D(R) &\approx 8i\pi^2 G_N L S_D \int_0^R dr \int_0^r dl \rho(r) \rho(l) rl \left[\sum_{n=1}^{\infty} \frac{1}{n^{2-D}} f(n, r, l) + \frac{1}{2} \lim_{n \rightarrow 0} \frac{1}{n^{2-D}} f(n, r, l) \right] \\ &= 8i\pi^2 G_N L S_D \int_0^R dr \int_0^r dl \rho(r) \rho(l) rl \left[\sum_{n=1}^{\infty} \frac{1}{n^{2-D}} f(n, r, l) + i\delta_{D,1} \frac{l}{L} \right]. \end{aligned} \quad (2.5)$$

For $D = 1$ this equation reproduces the exact result of the gravitational self-energy of the spherical body [11]. The first term in the kernel of Eq.(2.5) is the contribution of the tachyonic KK gravitons, while the second one is that of the ordinary massless gravitons. For $D \geq 2$ in this approximation the effect of the massless graviton exchange is condensed to the first term in the kernel of Eq.(2.5).

In order to be precise and concrete in our following arguments, we next set $\rho(r)$ to some typical densities and calculate the gravitational self-energies of the corresponding massive spherical bodies as follows.

2.1. The $\rho(r) = C_0$ (constant) case

We first set the density of the spherical body as $\rho(r) = C_0$, where C_0 is a positive constant. This is a simple and normal situation. Substituting this density into Eq.(2.5), we obtain

$$E_D(R) = 8i\pi^2 G_N L S_D C_0^2 \left[\sum_{n=1}^{\infty} \left\{ -\frac{iLR^3}{3n^{3-D}} - \frac{L^2 R^2}{2n^{4-D}} - \frac{L^2 R^2}{2n^{4-D}} \exp\left(i\frac{2R}{L}n\right) - \frac{iL^3 R}{n^{5-D}} \exp\left(i\frac{2R}{L}n\right) + \frac{L^4}{2n^{6-D}} \exp\left(i\frac{2R}{L}n\right) - \frac{L^4}{2n^{6-D}} \right\} + \delta_{D,1} \frac{iR^5}{15L} \right]. \quad (2.6)$$

So the real part of the self-energy becomes

$$\Re E_D(R) = 8\pi^2 G_N S_D C_0^2 L^3 \left[\frac{R^3}{3L} \zeta(3-D) + \frac{R^2}{2} \sum_{n=1}^{\infty} \frac{\sin \frac{2R}{L}n}{n^{4-D}} + LR \sum_{n=1}^{\infty} \frac{\cos \frac{2R}{L}n}{n^{5-D}} - \frac{L^2}{2} \sum_{n=1}^{\infty} \frac{\sin \frac{2R}{L}n}{n^{6-D}} - \delta_{D,1} \frac{R^5}{15L^3} \right], \quad (2.7)$$

and the imaginary part is

$$\Im E_D(R) = 8\pi^2 G_N S_D C_0^2 L^3 \left[-\frac{R^2}{2} \zeta(4-D) - \frac{L^2}{2} \zeta(6-D) - \frac{R^2}{2} \sum_{n=1}^{\infty} \frac{\cos \frac{2R}{L}n}{n^{4-D}} + LR \sum_{n=1}^{\infty} \frac{\sin \frac{2R}{L}n}{n^{5-D}} + \frac{L^2}{2} \sum_{n=1}^{\infty} \frac{\cos \frac{2R}{L}n}{n^{6-D}} \right], \quad (2.8)$$

where ζ is the zeta-function.

2.2. The $\rho(r) = \frac{C_1}{r}$ case

Next we consider the case of $\rho(r) = \frac{C_1}{r}$, where C_1 is a positive constant. This is also a physically normal setup. It is known that this produces interesting results especially for $D = 1$ [11]. Using this density we can calculate Eq.(2.5) as

$$E_D(R) = 8i\pi^2 G_N L S_D C_1^2 \left[\sum_{n=1}^{\infty} \left\{ -\frac{iLR}{n^{3-D}} - \frac{L^2}{2n^{4-D}} \exp\left(i\frac{2R}{L}n\right) + \frac{2L^2}{n^{4-D}} \exp\left(i\frac{R}{L}n\right) - \frac{3L^2}{2n^{4-D}} \right\} + i\delta_{D,1} \frac{R^3}{6L} \right]. \quad (2.9)$$

From this equation we obtain the real part of the self-energy as

$$\Re E_D(R) = 8\pi^2 G_N S_D C_1^2 L^3 \left[\frac{R}{L} \zeta(3-D) + \frac{1}{2} \sum_{n=1}^{\infty} \frac{\sin \frac{2R}{L}n}{n^{4-D}} - 2 \sum_{n=1}^{\infty} \frac{\sin \frac{R}{L}n}{n^{4-D}} - \delta_{D,1} \frac{R^3}{6L^3} \right], \quad (2.10)$$

and the imaginary part as

$$\Im E_D(R) = 8\pi^2 G_N S_D C_1^2 L^3 \left[-\frac{3}{2} \zeta(4-D) - \frac{1}{2} \sum_{n=1}^{\infty} \frac{\cos \frac{2R}{L}n}{n^{4-D}} + 2 \sum_{n=1}^{\infty} \frac{\cos \frac{R}{L}n}{n^{4-D}} \right]. \quad (2.11)$$

2.3. The $\rho(r) = \frac{C_2}{r^2}$ case

At last we let $\rho(r) = \frac{C_2}{r^2}$, where C_2 is a positive constant. This is a setup singular at the origin $r = 0$. From Eq.(2.5) the self energy of the spherical body is

$$E_D(R) = 8i\pi^2 G_N L S_D C_2^2 \left[\sum_{n=1}^{\infty} \frac{2}{n^{2-D}} \left\{ \sum_{q=0}^{\infty} \sum_{k=q}^{\infty} \frac{(-1)^q}{(2k+1)^2 (2q)!} \left(\frac{nR}{L}\right)^{2q} \exp\left(i\frac{R}{L}n\right) + i \sum_{q=0}^{\infty} \sum_{k=q+1}^{\infty} \frac{(-1)^{q+1}}{(2k+1)^2 (2q+1)!} \left(\frac{nR}{L}\right)^{2q+1} \exp\left(i\frac{R}{L}n\right) - \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \right\} + i\delta_{D,1} \frac{R}{L} \right]. \quad (2.12)$$

So the real part of the self-energy is

$$\Re E_D(R) = 8\pi^2 G_N S_D C_2^2 L \left[2 \sum_{q=0}^{\infty} \left\{ \sum_{k=q}^{\infty} \frac{(-1)^{q+1}}{(2k+1)^2 (2q)!} \left(\frac{R}{L}\right)^{2q} \sum_{n=1}^{\infty} \frac{\sin \frac{R}{L}n}{n^{2-D-2q}} + \sum_{k=q+1}^{\infty} \frac{(-1)^q}{(2k+1)^2 (2q+1)!} \left(\frac{R}{L}\right)^{2q+1} \sum_{n=1}^{\infty} \frac{\cos \frac{R}{L}n}{n^{1-D-2q}} \right\} - \delta_{D,1} \frac{R}{L} \right], \quad (2.13)$$

and the imaginary part is

$$\begin{aligned} \Im E_D(R) = 16\pi^2 G_N S_D C_2^2 L \left[\sum_{q=0}^{\infty} \left\{ \sum_{k=q}^{\infty} \frac{(-1)^q}{(2k+1)^2 (2q)!} \left(\frac{R}{L} \right)^{2q} \sum_{n=1}^{\infty} \frac{\cos \frac{R}{L} n}{n^{2-D-2q}} \right. \right. \\ \left. \left. + \sum_{k=q+1}^{\infty} \frac{(-1)^q}{(2k+1)^2 (2q+1)!} \left(\frac{R}{L} \right)^{2q+1} \sum_{n=1}^{\infty} \frac{\sin \frac{R}{L} n}{n^{1-D-2q}} \right\} - \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \zeta(2-D) \right]. \end{aligned} \quad (2.14)$$

3. The gravitational stability

As we have seen in the previous section, with the tachyon exchange potential Eq.(2.1) the gravitational self-energies of massive spherical bodies normally have imaginary parts, which inevitably lead to gravitational instabilities. On the other hand, as we have shown in our previous paper [11] for the case of $D = 1$, the spherical bodies which have just right mass densities with right discrete values of the radius such that the imaginary parts of the self-energies vanish, become stable.

In the following we shall discuss the gravitational stability of massive bodies with higher D extra timelike dimensions, $D \geq 1$.

3.1. The $\rho(r) = C_0$ case

We first consider the $\rho(r) = C_0$ case. For $D = 1$ we can calculate the imaginary part of the self-energy from Eq.(2.5) as

$$\Im E_1(R) = -8\pi^2 G_N S_1 C_0^2 L \int_0^R dr \int_0^r dl \, r l \log \left| \frac{\sin \frac{r+l}{2L}}{\sin \frac{r-l}{2L}} \right| \quad (3.1)$$

and its numerical result is presented in our previous paper [11], which says that, since $\Im E_1(R)$ does not vanish at any R , there is no stable radius of the spherical body.

Using Eqs.(2.8), (A.4), (A.5) and (A.8), for $D = 2$ the imaginary part of the self-energy at R in the region $0 < R < \pi L$ becomes

$$\Im E_2(R) = -8\pi^2 G_N S_2 C_0^2 L^3 \left(\frac{\pi}{6} \frac{R^3}{L} \right), \quad (3.2)$$

and it is proportional to $\sim L^2 R^3$.

Using Eqs.(2.8), (A.6), (A.7) and (A.10), for $D = 3$ we obtain the imaginary part of the self-energy at $0 < R < \pi L$ as

$$\Im E_3(R) = -8\pi^2 G_N S_3 C_0^2 L^3 \left[\frac{1}{2} R^2 \left(\zeta(1) - 1 + \log \frac{2R}{L} \right) + \mathcal{O}(R^4) \right], \quad (3.3)$$

which diverges since the zeta-function $\zeta(z)$ has a pole at $z = 1$. So the spherical body is unstable for $D = 3$.

For $D = 4$, using Eqs.(2.8), (A.8), (A.9) and (A.12), the imaginary part of the self-energy at $k\pi L < R < (k+1)\pi L$ ($k \in \{\mathbf{N}, 0\}$) turns out to be a step function given as

$$\begin{aligned}\Im E_4(R) &= -8\pi^2 G_N S_4 C_0^2 L^3 \left(\frac{R^2}{2} \zeta(0) + \frac{L^2}{2} \zeta(2) + \frac{1}{4} R^2 - \frac{\pi^2}{12} L^2 - \frac{\pi^2}{2} k(k+1) L^2 \right) \\ &= -8\pi^2 G_N S_4 C_0^2 L^3 \left(-\frac{\pi^2}{2} k(k+1) L^2 \right),\end{aligned}\quad (3.4)$$

where we have used $\zeta(0) = -\frac{1}{2}$ and $\zeta(2) = \frac{\pi^2}{6}$. Thus the spherical bodies with any radii $0 < R < \pi L$ ($k = 0$) are stable, while at $R > \pi L$ ($k \geq 1$) they are unstable.

Using Eqs.(2.8), (A.8), (A.9) and (A.12), for $D = 5$ we calculate the imaginary part of the self-energy at $0 < R < \pi L$ as

$$\Im E_5(R) = -8\pi^2 G_N S_5 C_0^2 L^3 \left[\frac{L^2}{2} \left(\zeta(1) - \frac{5}{4} + \log \frac{2R}{L} \right) + \mathcal{O}(R^6) \right], \quad (3.5)$$

and this equation diverges for the same reason as for $D = 3$.

For $D = 6$, using Eqs.(2.8), (A.12) and (A.15), the imaginary part of the self-energy at all R becomes

$$\Im E_6(R) = -8\pi^2 G_N S_6 C_0^2 L^3 \left(\frac{R^2}{2} \zeta(-2) + \frac{L^2}{2} \zeta(0) + \frac{L^2}{4} \right) = 0, \quad (3.6)$$

where $\zeta(-2) = 0$ and $\zeta(0) = -\frac{1}{2}$. Eq.(3.6) implies that the spherical bodies can be stable.

For $D = 2s + 6$ ($s \in \mathbf{N}$) Eq.(2.8) is

$$\begin{aligned}\Im E_{2s+6}(R) &= 8\pi^2 G_N S_{2s+6} C_0^2 L^3 \left(\frac{R^2}{2} \zeta(-2-2s) + \frac{L^2}{2} \zeta(-2s) + \frac{R^2}{2} \sum_{n=1}^{\infty} \frac{\cos \frac{2R}{L} n}{n^{-2-2s}} \right. \\ &\quad \left. - LR \sum_{n=1}^{\infty} \frac{\sin \frac{2R}{L} n}{n^{-1-2s}} - \frac{L^2}{2} \sum_{n=1}^{\infty} \frac{\cos \frac{2R}{L} n}{n^{-2s}} \right),\end{aligned}\quad (3.7)$$

and, substituting Eq.(A.15) into this equation, it becomes

$$\Im E_{2s+6}(R) = 8\pi^2 G_N S_{2s+6} C_0^2 L^3 \left(\frac{R^2}{2} \zeta(-2-2s) + \frac{L^2}{2} \zeta(-2s) \right). \quad (3.8)$$

Since $\zeta(-2t) = 0$ ($t \in \mathbf{N}$), from Eq.(3.8) we obtain the imaginary part of the self-energy at all R as

$$\Im E_{2s+6}(R) = 0, \quad (3.9)$$

and then the spherical bodies are stable.

3.2. The $\rho(r) = \frac{C_1}{r}$ case

Next we consider the $\rho(r) = \frac{C_1}{r}$ case. For $D = 1$ we know from our previous paper [11] that, since the imaginary part of the self-energy has a periodicity of R with a pitch $2\pi L$ expressed as

$$\Im E_1(2\pi Lk + c) = -8\pi^2 G_N S_1 C_1^2 L \int_0^c dr \int_0^r dl \log \frac{|\sin \frac{r+l}{2L}|}{\sin \frac{r-l}{2L}}, \quad k \in \{\mathbf{N}, 0\}, \quad 0 \leq c < 2\pi L, \quad (3.10)$$

the spherical bodies of the radii $R = 2\pi Lk$ become stable. More generally for all D Eq.(2.11) also has the same periodicity of R with a pitch $2\pi L$, so we substitute $R = 2\pi Lk$ into Eq.(2.11) and calculate it as

$$\Im E_D(2\pi Lk) = 8\pi^2 G_N S_D C_1^2 L^3 \left[-\frac{3}{2}\zeta(4-D) - \frac{1}{2}\zeta(4-D) + 2\zeta(4-D) \right] = 0. \quad (3.11)$$

Remarkably enough, the spherical bodies can be stable at the radii $R = 2\pi Lk$ for any number D of the extra timelike dimensions.

For $D = 2$ from Eqs.(2.11) and (A.8) we obtain the imaginary part of the self-energy at $0 < R < \pi L$ as

$$\Im E_2(R) = -8\pi^2 G_N S_2 C_1^2 L^3 \left(\frac{\pi R}{2L} \right), \quad (3.12)$$

where we have used $\zeta(2) = \frac{\pi^2}{6}$. Since Eq.(3.12) is not zero, these spherical bodies are unstable.

For $D = 3$, using Eq.(A.10), at $0 < R < \pi L$ Eq.(2.11) becomes

$$\Im E_3(R) = 8\pi^2 G_N S_3 C_1^2 L^3 \left[\frac{1}{2} \log 2 - \frac{3}{2} \log \frac{R}{L} - \frac{1}{480} \left(\frac{R}{L} \right)^4 - \cdots - \frac{3}{2} \zeta(1) \right], \quad (3.13)$$

where $\zeta(1)$ is a single pole, then the spherical bodies is not stable.

For $D = 4$ from Eqs.(2.11) and (A.12) we obtain the imaginary part of the self-energies at all R as

$$\Im E_4(R) = 8\pi^2 G_N S_4 C_1^2 L^3 \left(-\frac{3}{4} - \frac{3}{2} \zeta(0) \right) = 0, \quad (3.14)$$

where $\zeta(0) = -\frac{1}{2}$. Then from Eq.(3.14) the spherical bodies become stable.

For $D = 5$, using Eqs.(2.11) and (A.13), we calculate the imaginary part of the self-energy at $0 < R < \pi L$ as

$$\Im E_5(R) = 8\pi^2 G_N S_5 C_1^2 L^3 \left(-\frac{15}{8} \frac{L^2}{R^2} - \frac{1}{128} \frac{R^2}{L^2} - \cdots \right). \quad (3.15)$$

For $D = 2s + 4$ ($s \in \mathbf{N}$) Eq.(2.11) is

$$\Im E_{2s+4}(R) = 8\pi^2 G_N S_{2s+4} C_1^2 L^3 \left[-\frac{3}{2} \zeta(-2s) - \frac{1}{2} \sum_{n=1}^{\infty} \frac{\cos \frac{2R}{L} n}{n^{2s}} + 2 \sum_{n=1}^{\infty} \frac{\cos \frac{R}{L} n}{n^{2s}} \right], \quad (3.16)$$

and, substituting Eq.(A.15) into Eq.(3.16) we obtain

$$\Im E_{2s+4}(R) = 8\pi^2 G_N S_{2s+4} C_1^2 L^3 \left(-\frac{3}{2} \zeta(-2s) \right). \quad (3.17)$$

Since $\zeta(-2s) = 0$, the imaginary part of the self-energy at all R becomes

$$\Im E_{2s+4}(R) = 0, \quad (3.18)$$

and the spherical bodies then are stable.

3.3. The $\rho(r) = \frac{C_2}{r^2}$ case

Finally we consider the $\rho(r) = \frac{C_2}{r^2}$ case. For $D = 1$ Eq.(2.14) includes $\zeta(1)$. So $\Im E_1(R)$ becomes singular. Using Eqs.(A.12) and (A.15), for $D = 2$ at all R Eq.(2.14) becomes

$$\Im E_2(R) = 16\pi^2 G_2 S_2 C_2^2 L \left[\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \left(-\frac{1}{2} \right) - \zeta(0) \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \right] = 0, \quad (3.19)$$

where we have used $\zeta(0) = -\frac{1}{2}$. We find the spherical bodies to be stable from Eq.(3.19).

And also for $D = 2s + 2$ ($s \in \mathbf{N}$) from Eqs.(2.14) and (A.15) the imaginary part of the self-energy at all R becomes

$$\Im E_{2s+2}(R) = 0, \quad (3.20)$$

and then the spherical bodies are stable.

4. The screening effect

Now we consider the real part of the gravitational self-energy and find that for a certain number D of the extra timelike dimensions at a certain region of R the gravitational self-energy is screened.

4.1. The $\rho(r) = C_0$ case

At first we discuss the $\rho(r) = C_0$ case. For $D = 1$, as previously shown in Ref.[11], from Eq.(2.7) the real part of the self-energy becomes at $0 \leq c < \pi L$

$$\begin{aligned} \Re E_1(2\pi Lk + c) = & -\frac{8}{45}G_N S_1 C_0^2 L^2 \pi^4 (2k+1)k[30c^3 + 15(8k-1)\pi Lc^2 \\ & + 60k(2k-1)(\pi L)^2 c + (48k^3 - 24k^2 + 2k - 1)(\pi L)^3], \end{aligned} \quad (4.1)$$

where $k \in \{\mathbf{N}, 0\}$, and at $\pi L \leq c < 2\pi L$

$$\begin{aligned} \Re E_1(2\pi Lk + c) = & -\frac{8}{45}G_N S_1 C_0^2 L^2 \pi^4 (k+1)(2k+1)[30c^3 + 15(8k-3)\pi Lc^2 \\ & + 60k(2k-3)(\pi L)^2 c + (48k^3 - 72k^2 + 74k + 15)(\pi L)^3]. \end{aligned} \quad (4.2)$$

Eqs.(4.1) and (4.2) imply that the gravitational self-energies of the spherical bodies of radii $0 \leq R \leq \pi L$ are screened.

Now we concentrate on the spherical bodies of radii $0 < R < \pi L$. For $D = 2$, substituting Eqs.(A.3), (A.6) and (A.7) into Eq.(2.7), we obtain

$$\Re E_2(R) = 8\pi^2 G_N S_2 C_0^2 L^3 \left[\frac{R^3}{L} \left(\frac{1}{3}\zeta(1) + \frac{1}{3} \log \frac{2R}{L} - \frac{7}{9} \right) + \mathcal{O}(R^5) \right]. \quad (4.3)$$

Since Eq.(4.3) includes $\zeta(1)$, which is a pole, the real part of the self-energy diverges.

For $D = 3$, using Eqs.(A.5), (A.8) and (A.9), Eq.(2.7) becomes

$$\Re E_3(R) = 8\pi^2 G_N S_3 C_0^2 L^3 \left(\frac{R^3}{3L} \zeta(0) - \frac{1}{4} \pi R^2 + \frac{1}{6} \frac{R^3}{L} \right) = 8\pi^2 G_N S_3 C_0^2 L^3 \left(-\frac{\pi}{4} R^2 \right), \quad (4.4)$$

where $\zeta(0)$ is $-\frac{1}{2}$.

For $D = 4$ from Eqs.(A.7), (A.10) and (A.11) we calculate Eq.(2.7) as

$$\begin{aligned} \Re E_4(R) = & 8\pi^2 G_N S_4 C_0^2 L^3 \left(\frac{R^3}{3L} \zeta(-1) - \frac{3}{4} LR + \frac{1}{36} \frac{R^3}{L} - \frac{1}{900} \frac{R^5}{L^3} + \mathcal{O}(R^7) \right) \\ = & 8\pi^2 G_N S_4 C_0^2 L^3 \left(-\frac{3}{4} LR - \frac{1}{900} \frac{R^5}{L^3} + \mathcal{O}(R^7) \right), \end{aligned} \quad (4.5)$$

where $\zeta(-1) = -\frac{1}{12}$.

For $D = 5$ from Eqs.(2.7), (A.9), (A.12) and (A.15) the real part of the self-energy becomes

$$\begin{aligned} \Re E_5(R) = & 8\pi^2 G_N S_5 C_0^2 L^3 \left[\frac{R^3}{3L} \zeta(-2) - \frac{1}{2} LR - \frac{L^2}{4} \left(\pi - \frac{2R}{L} \right) \right] \\ = & 8\pi^2 G_N S_5 C_0^2 L^3 \left(-\frac{\pi}{4} L^2 \right) \end{aligned} \quad (4.6)$$

and it is a negative constant independent of R .

For $D = 6$, substituting Eqs.(A.11), (A.13) and (A.14) into Eq.(2.7), we obtain

$$\begin{aligned}\Re E_6(R) &= 8\pi^2 G_N S_6 C_0^2 L^3 \left(\frac{R^3}{3L} \zeta(-3) - \frac{5}{8} \frac{L^3}{R} - \frac{1}{360} \frac{R^3}{L} + \dots \right) \\ &= 8\pi^2 G_N S_6 C_0^2 L^3 \left(-\frac{5}{8} \frac{L^3}{R} + \mathcal{O}(R^5) \right),\end{aligned}\quad (4.7)$$

where $\zeta(-3) = \frac{1}{120}$ has been used. The real part of the self-energy Eq.(4.7) diverges for $R \rightarrow 0$.

For $D = 2s + 5$ ($s \in \mathbf{N}$) we have an interesting result at all R . Eq.(2.7) is then given as

$$\begin{aligned}\Re E_{2s+5}(R) &= 8\pi^2 G_N S_{2s+5} C_0^2 L^3 \left[\frac{R^3}{3L} \zeta(-2-2s) + \frac{R^2}{2} \sum_{n=1}^{\infty} \frac{\sin \frac{2R}{L} n}{n^{-1-2s}} \right. \\ &\quad \left. + LR \sum_{n=1}^{\infty} \frac{\cos \frac{2R}{L} n}{n^{-2s}} - \frac{L^2}{2} \sum_{n=1}^{\infty} \frac{\sin \frac{2R}{L} n}{n^{1-2s}} \right].\end{aligned}\quad (4.8)$$

Since $\zeta(-2-2s)$ are zero, Eq.(4.8) becomes by use of Eq.(A.15)

$$\Re E_{2s+5}(R) = 0. \quad (4.9)$$

The vanishing of the real part of the self-energy, Eq.(4.9), at all R leads to the screening effect.

4.2. The $\rho(r) = \frac{C_1}{r}$ case

Next we consider the $\rho(r) = \frac{C_1}{r}$ case. For $D = 1$ from (2.10) the real part of the self-energy, which is already given in Ref.[11], is

$$\Re E_1(2\pi Lk + c) = -16\pi^3 G_N S_1 C_1^2 Lk [3c^2 + 6\pi Lkc + (4k^2 - 1)\pi^2 L^2] \quad (4.10)$$

at $0 \leq c < \pi L$ and

$$\Re E_1(2\pi Lk + c) = -16\pi^3 G_N S_1 C_1^2 L(k+1) [3c^2 + 6(k-1)\pi Lc + (4k^2 - 4k + 3)\pi^2 L^2] \quad (4.11)$$

at $\pi L \leq c < 2\pi L$, where $k \in \{\mathbf{N}, 0\}$. So the real part of the gravitational self-energy with $0 < R < \pi L$ is screened.

We now pay attention to the spherical bodies of radii $0 < R < \pi L$ for $D \geq 2$. Substituting Eq.(A.7) into Eq.(2.10), the real part of the self-energy for $D = 2$ becomes

$$\Re E_2(R) = 8\pi^2 G_N S_2 C_1^2 L^3 \left[\frac{R}{L} \left(\zeta(1) + \log \frac{R}{2L} - 1 \right) + \frac{1}{36} \frac{R^3}{L^3} + \mathcal{O}(R^5) \right], \quad (4.12)$$

which can not converge because $\zeta(1)$ is a pole.

For $D = 3$ from Eqs.(2.10) and (A.15) the real part of the self-energy is

$$\begin{aligned} \Re E_3(R) &= 8\pi^2 G_N S_3 C_1^2 L^3 \left[\frac{R}{L} \zeta(0) + \frac{1}{4} \left(\pi - \frac{2R}{L} \right) - \left(\pi - \frac{R}{L} \right) \right] \\ &= 8\pi^2 G_N S_3 C_1^2 L^3 \left(-\frac{3}{4} \pi \right), \end{aligned} \quad (4.13)$$

where $\zeta(0) = -\frac{1}{2}$ and it is a negative constant.

For $D = 4$ from Eq.(A.11) the real part of the self-energy Eq.(2.10) becomes

$$\Re E_4(R) = 8\pi^2 G_N S_4 C_1^2 L^3 \left(-\frac{7}{4} \frac{L}{R} - \frac{1}{360} \frac{R^3}{L^3} + \mathcal{O}(R^5) \right), \quad (4.14)$$

where we have used $\zeta(-1) = -\frac{1}{12}$. The result diverges when $R \rightarrow 0$.

For $D = 2s + 3$ ($s \in \mathbf{N}$) we also have a characteristic result at any R . From Eq.(2.10) we obtain

$$\Re E_{2s+3}(R) = 8\pi^2 G_N S_{2s+3} C_1^2 L^3 \left[\frac{R}{L} \zeta(-2s) + \frac{1}{2} \sum_{n=1}^{\infty} \frac{\sin \frac{2R}{L} n}{n^{1-2s}} - 2 \sum_{n=1}^{\infty} \frac{\sin \frac{R}{L} n}{n^{1-2s}} \right]. \quad (4.15)$$

And, since we know Eq.(A.15) and $\zeta(-2s) = 0$ to hold, Eq.(4.15) becomes

$$\Re E_{2s+3}(R) = 0. \quad (4.16)$$

So Eq.(4.16) leads to the screening of the gravitational self-energy at all R .

4.3. The $\rho(r) = \frac{C_2}{r^2}$ case

We consider the $\rho(r) = \frac{C_2}{r^2}$ case. For $D = 1$ at $0 < R < \pi L$ from Eqs.(2.13), (A.9), (A.12) and (A.15) we obtain with the formula $\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}$

$$\begin{aligned} \Re E_1(R) &= 8\pi^2 G_N S_1 C_2^2 L \left[2 \left(-\frac{1}{2} \pi \sum_{k=1}^{\infty} \frac{1}{(2k+1)^2} - \frac{\pi}{2} + \frac{1}{2} \frac{R}{L} \right) - \frac{R}{L} \right] \\ &= 8\pi^2 G_N S_1 C_2^2 L \left(-\frac{\pi^3}{8} \right), \end{aligned} \quad (4.17)$$

which shows that the real part of the self-energy becomes a negative constant.

For $D = 2s + 1$ ($s \in \mathbf{N}$) Eq.(2.13) becomes

$$\begin{aligned} \Re E_{2s+1}(R) = & 8\pi^2 G_N S_{2s+1} C_2^2 L \left[2 \sum_{q=0}^{\infty} \left\{ \sum_{k=q}^{\infty} \frac{(-1)^{q+1}}{(2k+1)^2 (2q)!} \left(\frac{R}{L} \right)^{2q} \sum_{n=1}^{\infty} \frac{\sin \frac{R}{L} n}{n^{1-2s-2q}} \right. \right. \\ & \left. \left. + \sum_{k=q+1}^{\infty} \frac{(-1)^q}{(2k+1)^2 (2q+1)!} \left(\frac{R}{L} \right)^{2q+1} \sum_{n=1}^{\infty} \frac{\cos \frac{R}{L} n}{n^{-2s-2q}} \right\} \right]. \end{aligned} \quad (4.18)$$

Since from Eq.(A.15) the equalities $\sum_{n=1}^{\infty} \frac{\sin \frac{R}{L} n}{n^{1-2s-2q}} = \sum_{n=1}^{\infty} \frac{\cos \frac{R}{L} n}{n^{-2s-2q}} = 0$ hold, we can calculate the real part of the self-energy (4.18) at any R as

$$\Re E_{2s+1}(R) = 0. \quad (4.19)$$

Then the gravitational self-energy is screened at all R .

5. The Newton constants in D extra timelike dimensions

The correlation between the complexity of the Newtonian potential and the number q of extra times has been discussed in Ref.[10] from somewhat different perspective. It is pointed out there that the Newtonian potential $m^2 V(d)$ between two point-like masses m which are localized at a particular time moment $\tau = 0$ in q extra times at $d \ll L$ distance apart becomes pure imaginary for odd q like

$$m^2 V(d) \sim (i)^q \frac{m^2}{M_{Pl(4+q)}^{2+q}} \frac{1}{d^{1+q}}$$

for the static case, meaning that the Newtonian potential is screened at $d \ll L$ distances. Our work is expected to shed some light on this line of investigation as well.

In fact, in this section we shall derive the exact relationship between the Newton constants $\hat{G}_{N(4+D)}$ in the D extra timelike dimensions and the ordinary 4 dimensional Newton constant $G_N = \hat{G}_{N(4)}$ with $D = 0$, which are defined by the attractive force laws between two mass points m_1, m_2 at distance d

$$\begin{aligned} \hat{F}_{(4+D)}(d) &= -\hat{G}_{N(4+D)} \frac{m_1 m_2}{d^{2+D}}, \\ \hat{F}_{(4)}(d) &= -G_N \frac{m_1 m_2}{d^2}. \end{aligned} \quad (5.1)$$

Now we start with the gravitational potential Eq.(2.2) between two unit mass points. As we have done in Section 2, we perform the replacement in Eq.(2.2) to transform the infinite summation to the integration as

$$\begin{aligned}
m^2 V(d) &\approx -G_N \frac{m^2}{d} \int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_D \exp \left(i \frac{d}{L} \sqrt{x_1^2 + \cdots + x_D^2} \right) \\
&= -G_N \frac{m^2}{d} \int d\Omega_{D-1} \int_0^{\infty} dt t^{D-1} \exp \left(i \frac{d}{L} t \right) \\
&= -G_N \frac{m^2}{d} S_D \int_0^{\infty} dt t^{D-1} \exp \left(i \frac{d}{L} t \right), \tag{5.2}
\end{aligned}$$

where $S_D = \frac{2\pi^{\frac{D}{2}}}{\Gamma(\frac{D}{2})}$. Note that this equation becomes exact when $d \ll L$, that is, for the limit $d/L \rightarrow 0$. By deforming the integration contour through the analytic continuation method, we finally obtain with the D dimensional timelike volume $V_D = (2\pi L)^D$

$$\begin{aligned}
m^2 V(d) &= -(i)^D G_N S_D \Gamma(D) L^D \frac{m^2}{d^{1+D}} \\
&= -(i)^D G_N \frac{S_D \Gamma(D) V_D}{(2\pi)^D} \frac{m^2}{d^{1+D}} \\
&= -\frac{\hat{G}_{N(4+D)}}{1+D} \frac{m^2}{d^{1+D}}, \tag{5.3}
\end{aligned}$$

which implies that the following relationship holds between the Newton constants for the D extra timelike dimensions and the Newton constant of our world

$$\hat{G}_{N(4+D)} = (i)^D \frac{4\pi V_D}{S_{(3+D)}} G_N. \tag{5.4}$$

Thus we find that, since G_N is real, the Newton constant $\hat{G}_{N(4+D)}$ of the full $4+D$ dimensional theories with the D extra times becomes pure imaginary for odd D , while for even D it is pure real.

6. Conclusions

We discussed the D extra timelike dimensions and compactified them on the circles of the radius L . Then the tachyonic Kaluza-Klein modes are induced. We let only the gravitons propagate in the D extra timelike dimensions. And we calculated the gravitational self-energies of spherical bodies of radii R in the fairly good approximation. Note that for $D = 1$ the results are exact. The tachyonic KK gravitons give rise to the imaginary parts of

the self-energies, which leads to the instability of the spherical bodies. In some dimensions the contribution of the ordinary massless gravitons to the self-energies are canceled out by the one of the tachyonic KK gravitons, and the self-energies of the spherical bodies of certain radii with certain mass densities are screened.

We considered the imaginary parts of the self-energies of spherical bodies with the three typical and spherically symmetric mass densities and discussed their stability. At first we set the density $\rho(r) = C_0$. From Eqs.(3.6) and (3.9) the imaginary parts of the self-energies of the spherical bodies which have any radii R vanish for $D = 2s + 4$ ($s \in \mathbf{N}$) then the spherical bodies are stabilized. And also for $D = 4$ the imaginary part of the self-energy becomes zero at each value R for the range $0 < R < \pi L$ from Eq.(3.4), so the spherical body is stable. Since for $D = 3$ and 5 Eqs.(3.3) and (3.5) involve $\zeta(1)$, which is a pole, the imaginary parts of the self-energies diverge at $0 < R < \pi L$. Next we let the density $\rho(r) = \frac{C_1}{r}$. This has the interesting features. Eq.(3.11) shows that the spherical bodies which have critical radii $R = 2\pi Lk$ ($k \in \{0, \mathbf{N}\}$) are stable for any dimension D . And from Eqs.(3.14) and (3.18) for $D = 2s + 2$ ($s \in \mathbf{N}$), the imaginary parts of the self-energies become identically zero at all R , so the corresponding spherical bodies with any value of the radii R become stable. At last we adopted $\rho(r) = \frac{C_2}{r^2}$ as the density. For $D = 2s$ ($s \in \mathbf{N}$), from Eqs.(3.19) and (3.20) the imaginary parts of the gravitational self-energies again vanish at all R , then the spherical bodies of any radii R are again stable.

And we discussed the screening effects due to the tachyonic KK gravitons which are signaled by the vanishing of the real parts of the self-energies. When $\rho(r)$ has the constant value C_0 , at the region $0 < R < \pi L$ the gravitational force is screened for $D = 1$ from Eq.(4.1). At that region of R from Eq.(4.3) the real part of the self-energy for $D = 2$ diverges because of the pole of $\zeta(1)$ and from Eq.(4.6) the one for $D = 5$ becomes a negative constant independent of R . On the other hand, for $D = 2s + 5$ ($s \in \mathbf{N}$), from Eq.(4.9) we can say that the gravitational force is screened at any R . We next considered the $\rho(r) = \frac{C_1}{r}$ case. Eq.(4.16) implies that for $D = 2s + 3$ ($s \in \mathbf{N}$) the gravitational forces are again screened at all R . At $0 < R < \pi L$ from Eq.(4.10) the real part of the gravitational self-energy for $D = 1$ vanishes, resulting in the screening of the gravitational force. At the same region from Eq.(4.12) the real part of the self-energy for $D = 2$ becomes a pole and from Eq.(4.13) the one for $D = 3$ is a negative constant, so we do not have a screening for these cases. Lastly we set $\rho(r) = \frac{C_2}{r^2}$. From Eq.(4.17) the real part of the self-energy becomes a negative constant for $D = 1$, resulting in no screening, while from Eq.(4.19) the

gravitational forces are screened for $D = 2s + 1$ ($s \in \mathbf{N}$) at all R due to the vanishing of the real parts of the corresponding self-energies.

On the last choice of $\rho(r) = \frac{C_2}{r^2}$ we have a comment in order for the case $D = 1$ of the extra timelike dimension. This choice of $\rho(r)$ is very singular and causes divergences in the exact formula of the gravitational self-energy for $D = 1$ presented in our previous paper [11], thus ruining the general statement that for $D = 1$ the gravitational force is screened for any spherical mass density $\rho(r)$ at $0 < R < \pi L$. Nevertheless, the claim in the statement remains to be generically valid for any $\rho(r)$ of reasonably mild analytic property.

We discussed the remarkable correlation between the number D of the extra times and the complexity of the Newtonian potential. In particular, we derived the exact relationship between the ordinary Newton constant G_N in our 4 dimensions and the Newton constant $\hat{G}_{N(4+D)}$ of the full $4 + D$ dimensional spacetime with the D extra times.

In our whole investigations we replaced the infinite summation (2.3) with the integration (2.4). This is the fairly good approximation in general and the replacement becomes exact for $l, r \ll L$, $0 \leq l, r \leq R$, or equivalently for $R \ll L$. And by transforming the integration back to the infinite summation (2.5) again, that approximation recovers a reasonable accuracy. In fact for $D = 1$ Eq.(2.5) is the exact result. At the region $R \ll L$ our consideration about the gravitational stabilities and screening effects can be regarded as precise, but at the other region it may be less reliable. This implies that at least we could say that if the extra timelike dimensions exist and the spherically symmetric body with a certain density is stable, the scale of the extra dimensions must be sufficiently larger than the one of the spherical body.

We studied the relations among the extra timelike dimensions, the stability of the spherical bodies and the screening effects of the gravitational force. These relations may give us some useful means for determining the size of particles or universes and for constructing some theories which include gravity.

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Appendix A. Infinite summations

The following infinite summations are known at $0 < x < 2\pi$ to hold:

$$\sum_{n=0}^{\infty} \frac{\sin nx}{n^5} = \frac{\pi^4}{90}x - \frac{\pi^2}{36}x^3 + \frac{\pi}{48}x^4 - \frac{1}{240}x^5, \quad (\text{A.1})$$

$$\sum_{n=0}^{\infty} \frac{\cos nx}{n^5} = \zeta(5) - \frac{\zeta(3)}{2}x^2 - \frac{1}{24}x^4 \log x + \frac{25}{288}x^4 + \frac{1}{8640}x^6 + \dots, \quad (\text{A.2})$$

$$\sum_{n=0}^{\infty} \frac{\sin nx}{n^4} = \zeta(3)x + \frac{1}{6}x^3 \log x - \frac{11}{36}x^3 - \frac{1}{1440}x^5 - \dots, \quad (\text{A.3})$$

$$\sum_{n=0}^{\infty} \frac{\cos nx}{n^4} = \frac{\pi^4}{90} - \frac{\pi^2}{12}x^2 + \frac{\pi}{12}x^3 - \frac{1}{48}x^4, \quad (\text{A.4})$$

$$\sum_{n=0}^{\infty} \frac{\sin nx}{n^3} = \frac{\pi^2}{6}x - \frac{\pi}{4}x^2 + \frac{1}{12}x^3, \quad (\text{A.5})$$

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{\cos nx}{n^3} &= \zeta(3) + \frac{x^2}{2} \log 2 + \int_0^x (x-t) \log \left(\sin \frac{t}{2} \right) dt \\ &= \zeta(3) + \frac{1}{2}x^2 \log x - \frac{3}{4}x^2 - \frac{1}{288}x^4 - \frac{1}{86400}x^6 - \dots, \end{aligned} \quad (\text{A.6})$$

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{\sin nx}{n^2} &= -x \log 2 - \int_0^x \log \left(\sin \frac{t}{2} \right) dt \\ &= -x \log x + x + \frac{1}{72}x^3 + \frac{1}{14400}x^5 + \dots, \end{aligned} \quad (\text{A.7})$$

$$\sum_{n=0}^{\infty} \frac{\cos nx}{n^2} = \frac{\pi^2}{6} - \frac{\pi}{2}x + \frac{1}{4}x^2, \quad (\text{A.8})$$

$$\sum_{n=0}^{\infty} \frac{\sin nx}{n} = \frac{\pi}{2} - \frac{1}{2}x, \quad (\text{A.9})$$

$$\sum_{n=0}^{\infty} \frac{\cos nx}{n} = -\log \left(2 \sin \frac{x}{2} \right) = -\log x + \frac{1}{24}x^2 + \frac{1}{2880}x^4 + \dots, \quad (\text{A.10})$$

$$\sum_{n=0}^{\infty} \sin nx = \frac{1}{x} - \frac{x}{12} - \frac{x^3}{720} + \dots, \quad (\text{A.11})$$

$$\sum_{n=0}^{\infty} \cos nx = -\frac{1}{2}, \quad (\text{A.12})$$

$$\sum_{n=0}^{\infty} n \cos nx = -\frac{1}{x^2} - \frac{1}{12} - \frac{1}{240}x^2 + \dots, \quad (\text{A.13})$$

$$\sum_{n=0}^{\infty} n^2 \sin nx = -\frac{2}{x^3} + \frac{x}{120} + \cdots, \quad (\text{A.14})$$

$$\sum_{n=0}^{\infty} n^{2p} \cos nx = \sum_{n=0}^{\infty} n^{2p-1} \sin nx = 0, \quad p \in \mathbf{N}. \quad (\text{A.15})$$

Eqs.(A.11)-(A.15) can be obtained by use of the equality (15) in page 30 of Ref. [12].

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